CONDENSED LESSON

3.1 Duplicating Segments and Angles

In this lesson, you

- Learn what it means to create a **geometric construction**
- **Duplicate a segment** by using a straightedge and a compass and by using patty paper
- **Duplicate an angle** by using a straightedge and a compass and by using patty paper

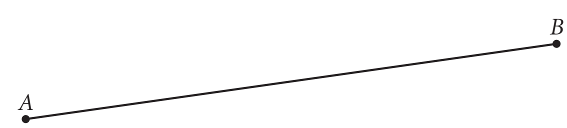
In geometry, there are several methods for creating a figure.

- You can *sketch* a figure without using geometry tools. Make a sketch when exact measurements are not important.
- You can *draw* a figure using measuring tools, such as a protractor and a ruler. Make a drawing when it is important for lengths and angle measures to be precise.
- You can *construct* a figure using a compass and straightedge. When you make a construction, do *not* use your measuring tools. Compass-and-straightedge constructions allow you to accurately draw congruent segments and angles, segment and angle bisectors, and parallel and perpendicular lines.
- You can also *construct* a figure using patty paper and a straightedge. As with compass-and-straightedge constructions, patty-paper constructions do not use measuring tools.

In this lesson, you will focus on constructions. You can read about the history of constructions in the lesson introduction in your book.

Investigation 1: Copying a Segment

In this investigation, you will copy this segment using only a compass and straightedge. When you construct a figure, you may use a ruler as a straightedge, but *not* as a measuring tool.



Draw a ray on your paper that extends longer than $\overline{AB}$. Label the endpoint of the ray point C. Now, think about how you can use *only* your compass to create a segment, $\overline{CD}$, that is the same length as $\overline{AB}$. Try constructing $\overline{CD}$ on your own before looking at Step 1 of the investigation in your book. You can use a ruler to check that $\overline{AB} \cong \overline{CD}$.

Step 1 shows the three stages involved in duplicating segment AB. The stages are described below.

Stage 1: Draw a ray that extends longer than $\overline{AB}$ and label the endpoint C.

Stage 2: Put the sharp end of your compass on point A. Open the compass until the other end touches point B, and make an arc.

Stage 3: *Without changing the opening of your compass*, put the sharp end of your compass on point C and make an arc on the ray. Label the point where the arc intersects the ray point D. Segment CD is congruent to segment AB.

(continued)

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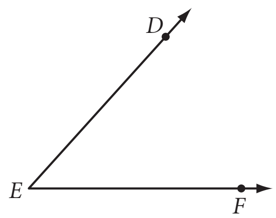
To duplicate $\overline{AB}$ using patty paper, simply place the patty paper over the segment and trace it, using a straightedge to ensure the tracing is straight.

Investigation 2: Copying an Angle

In this investigation, you will copy this angle using a compass and straightedge.



Construct a ray that extends longer than a side of $\angle DEF$. Label the endpoint of the ray G. This ray will be one side of the duplicate angle. Try to figure out how to duplicate $\angle DEF$ on your own before looking at Step 1 in your book. You can use a protractor to check that the angles are congruent.

Step 1 shows the first two stages involved in duplicating $\angle DEF$. The stages are described below.

Stage 1: Use your compass to construct an arc with its center at point E. The arc should intersect both sides of the angle. *Without changing the opening of your compass,* make an arc centered at point G.

Stage 2: On $\angle DEF$, put the sharp end of your compass on the point where the arc intersects $\overrightarrow{EF}$. Adjust the opening so that the other end touches the point where the arc intersects $\overrightarrow{ED}$, and make an arc. *Without changing the opening of your compass,* put the sharp end of your compass on the point where the arc intersects the ray with endpoint G and make an arc that intersects the original arc.

To finish the construction, draw a ray from point G through the point where the two arcs intersect. Use a protractor to verify that $\angle G$ is congruent to $\angle DEF$.

Practice duplicating other angles until you are sure you understand the steps. Be sure to try duplicating obtuse angles as well as acute angles.

Now, try to duplicate $\angle DEF$ using patty paper instead of a compass.

Write a summary of the construction methods you learned in this lesson.

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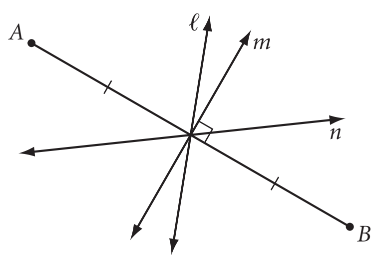
Constructing Perpendicular Bisectors

In this lesson, you

- Construct the **perpendicular bisector** of a segment using patty paper and using a compass and straightedge
- Complete the **Perpendicular Bisector Conjecture**
- Learn about **medians** and **midsegments** of triangles

A **segment bisector** is a line, ray, or segment that passes through the midpoint of the segment. A line that passes through the midpoint of a segment and that is perpendicular to the segment is the **perpendicular bisector** of the segment. A segment in a plane has an infinite number of bisectors, but it has *only one* perpendicular bisector.



Lines ℓ, m, and n bisect $\overline{AB}$.
Line m is the perpendicular bisector of $\overline{AB}$.

Investigation 1: Finding the Right Bisector

Follow Steps 1–3 in your book to construct a perpendicular bisector of segment PQ using patty paper.

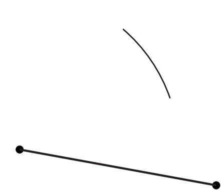
Place three points—A, B, and C—on the perpendicular bisector, and use your compass to compare the distances PA and QA, PB and QB, and PC and QC. In each case, you should find that the distances are equal. These findings lead to the following conjecture.

Perpendicular Bisector Conjecture If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints. **C-5**

Is the converse of this statement also true? That is, if a point is equidistant from the endpoints of a segment, is it on the segment's perpendicular bisector? If the converse is true, then locating two such points can help you locate the perpendicular bisector.

Investigation 2: Right Down the Middle

In this investigation, you will use a compass and straightedge to construct the perpendicular bisector of a segment. First, draw a line segment. Then, follow the steps below.



Adjust your compass so that the opening is more than half the length of the segment. Using one endpoint as the center, make an arc on one side of the segment.

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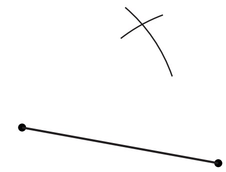
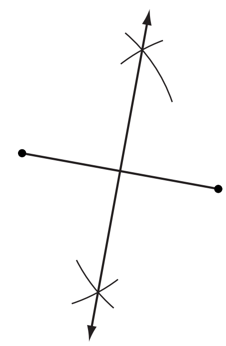
Without changing the opening of your compass, put the sharp end of your compass on the other endpoint and make an arc intersecting the first arc.



The point where the arcs intersect is equidistant from the two endpoints. Follow the same steps to locate another such point on the other side of the segment. Then, draw a line through the two points.

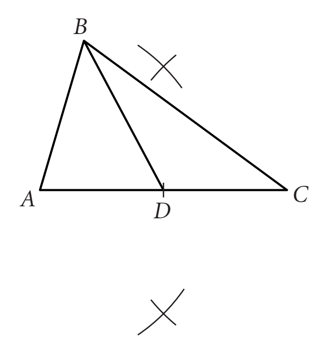


The line you drew is the perpendicular bisector of the segment. You can check this by folding the segment so that the endpoints coincide (as you did in Investigation 1). The line should fall on the crease of the paper.

The construction you did in this investigation demonstrates the conjecture below.

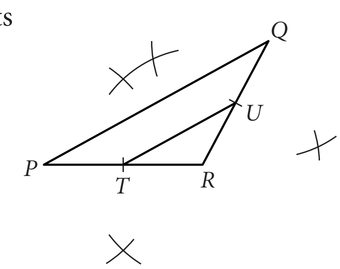
Converse of the Perpendicular Bisector Conjecture If a point is equidistant from the endpoints of a segment, then it is on the perpendicular bisector of the segment. **C-6**

Now that you know how to construct a perpendicular bisector, you can locate the midpoint of any segment. This allows you to construct two special types of segments related to triangles: medians and midsegments.



A **median** is a segment that connects a vertex of a triangle to the midpoint of the opposite side. To construct the median from vertex *B*, use the perpendicular bisector construction to locate the midpoint of $\overline{AC}$. Then, connect vertex *B* to this point.

A **midsegment** is a segment that connects the midpoints of two sides of a triangle. To construct a midsegment from $\overline{PR}$ to $\overline{QR}$, use the perpendicular bisector construction to locate midpoints of $\overline{PR}$ and $\overline{QR}$. Then, connect the midpoints.



Write a summary of the construction methods you learned in this lesson.

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CONDENSED LESSON

3.3 Constructing Perpendiculars to a Line

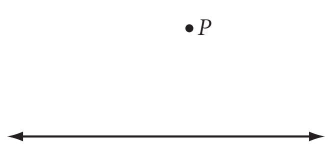
In this lesson, you

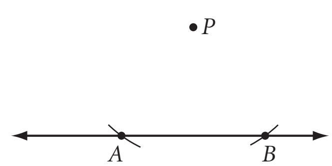
- Construct the **perpendicular** to a line from a point not on the line
- Complete the **Shortest Distance Conjecture**
- Learn about **altitudes** of triangles

In Lesson 3.2, you learned to construct the perpendicular bisector of a segment. In this lesson, you will use what you learned to construct the perpendicular to a line from a point not on the line.

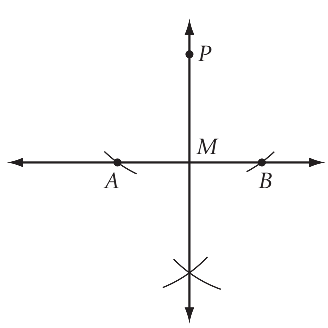
Investigation 1: Finding the Right Line

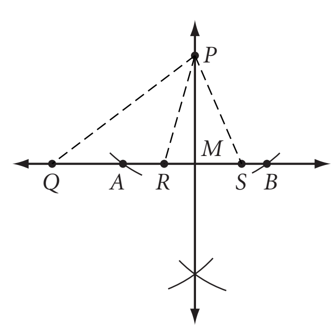
Draw a line and a point labeled *P* that is not on the line. With the sharp end of your compass at point *P*, make two arcs on the line. Label the intersection points *A* and *B*.





Note that $PA = PB$, so point *P* is on the perpendicular bisector of $\overline{AB}$. Use the construction you learned in Lesson 3.2 to construct the perpendicular bisector of $\overline{AB}$. Label the intersection point *M*. You have now constructed a perpendicular to a line from a point not on the line. Now choose any three points on $\overleftrightarrow{AB}$ and label them *Q*, *R*, and *S*. Measure *PQ*, *PR*, *PS*, and *PM*. Which distance is shortest?





Your observations should lead to this conjecture.

Shortest Distance Conjecture The shortest distance from a point to a line is measured along the perpendicular segment from the point to the line. **C-7**

In the next investigation, you will use patty paper to create a perpendicular from a point to a line.

(continued)

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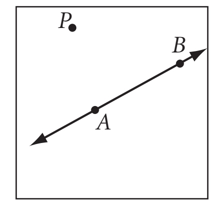
Investigation 2: Patty-Paper Perpendiculars

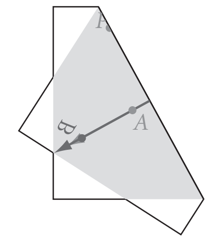
On a piece of patty paper, draw a line $\overleftrightarrow{AB}$ and a point P that is not on $\overleftrightarrow{AB}$.

Fold the line onto itself. Slide the layers of paper (keeping $\overleftrightarrow{AB}$ aligned with itself) until point P is on the fold.

Crease the paper, open it up, and draw a line on the crease. The line is the perpendicular to $\overleftrightarrow{AB}$ through point P. (Why?)



Step 1

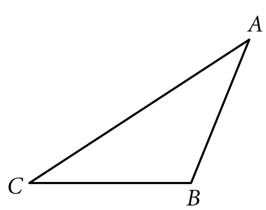


Step 2

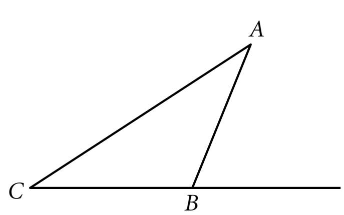
Constructing a perpendicular from a point to a line allows you to find the distance from the point to the line, which is defined as "The **distance from a point to a line** is the length of the perpendicular segment from the point to the line."

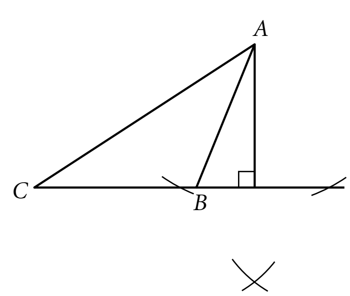
The **altitude** of a triangle is a perpendicular segment from a vertex of a triangle to the line containing the opposite side. The length of this segment is the height of the triangle. The illustrations on page 154 of your book show that an altitude can be inside or outside the triangle, or it can be one of the triangle's sides. A triangle has three different altitudes, so it has three different heights.

EXAMPLE Construct the altitude from vertex A of this triangle to side $\overleftrightarrow{CB}$.



▶ ***Solution*** Extend side $\overleftrightarrow{CB}$ and construct a perpendicular segment from point A to $\overleftrightarrow{CB}$.





Write a summary of the construction methods you learned in this lesson.

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CONDENSED LESSON

3.4 Constructing Angle Bisectors

In this lesson, you

- Construct an **angle bisector** using patty paper and using a compass
- Complete the **Angle Bisector Conjecture**

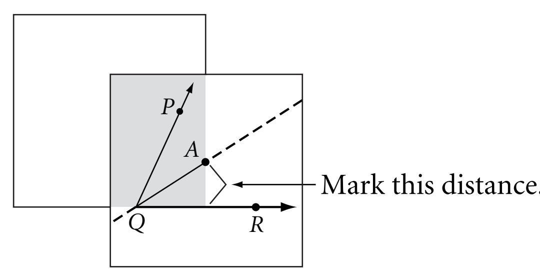
An **angle bisector** is a ray that divides an angle into two congruent angles. You can also refer to a segment as an angle bisector if the segment lies on the ray.

Investigation 1: Angle Bisecting by Folding

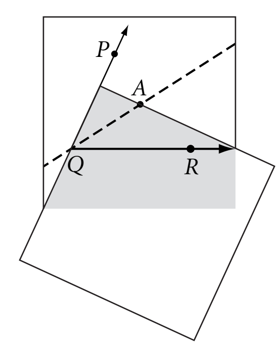
Follow Steps 1–3 in your book to construct the bisector of acute $\angle PQR$ using patty paper. You can tell the ray you construct is the angle bisector because the fold forms two angles that coincide.

Now, construct the bisector of an obtuse angle. Can you use the same method you used to bisect the acute angle?

Does every angle have a bisector? Is it possible for an angle to have more than one bisector? If you are not sure, experiment until you think you know the answers to both of these questions.



Look at the angles you bisected. Do you see a relationship between the points on the angle bisector and the sides of the angle? Choose one of the bisected angles. Choose any point on the bisector and label it *A*. Compare the distances from *A* to each of the two sides. (Remember that "distance" means *shortest* distance.) To do this, you can place one edge of a second piece of patty paper on one side of the angle. Slide the edge of the patty paper along the side of the angle until an adjacent perpendicular side of the patty paper passes through the point. Mark this distance on the patty paper.



Compare this distance with the distance to the other side of the angle by repeating the process on the other ray.

Your observations should lead to this conjecture.

Angle Bisector Conjecture If a point is on the bisector of an angle, then it is equidistant from the sides of the angle. **C-8**

(continued)

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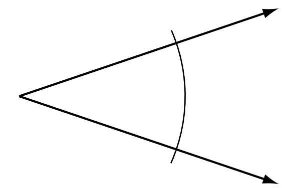
Investigation 2: Angle Bisecting with Compass

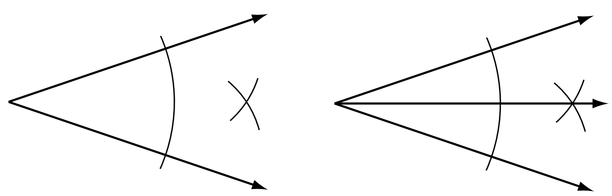
You can also construct an angle bisector using a compass and straightedge.

Draw an angle. To start the construction, draw an arc centered at the vertex of the angle that crosses both sides of the angle.



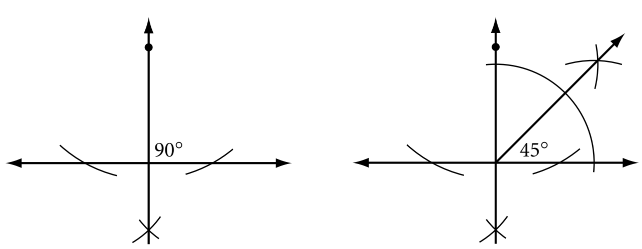
Try to complete the construction on your own before reading the following text. Don't be afraid to experiment. If you make a mistake, you can always start over. When you think you have constructed an angle bisector, fold your paper to check whether the ray you constructed is actually the bisector.

Constructing the angle bisector: Adjust your compass so that the opening is larger than half the length of the arc. Place the sharp end of your compass on one of the points where the arc intersects the angle, and make an arc. *Without changing the opening of your compass,* repeat this process with the other point of intersection. Draw the ray from the vertex of the angle to the point where the two small arcs intersect.



EXAMPLE Construct an angle with a measure of exactly 45° using only a compass and straightedge.

▶ ***Solution*** Construct a 90° angle by constructing the perpendicular to a line from a point not on the line. (Look back at Lesson 3.3 if you need to review this construction.) Then, use the angle-bisector construction you learned in this lesson to bisect the 90° angle.



Write a summary of the construction methods you learned in this lesson.

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CONDENSED LESSON

3.5 Constructing Parallel Lines

In this lesson, you

- Construct **parallel lines** using patty paper

As you learned in Chapter 1, **parallel lines** are lines that lie in the same plane and do not intersect. So, any two points on one parallel line will be equidistant from the other line. You can use this idea to construct a line parallel to a given line.

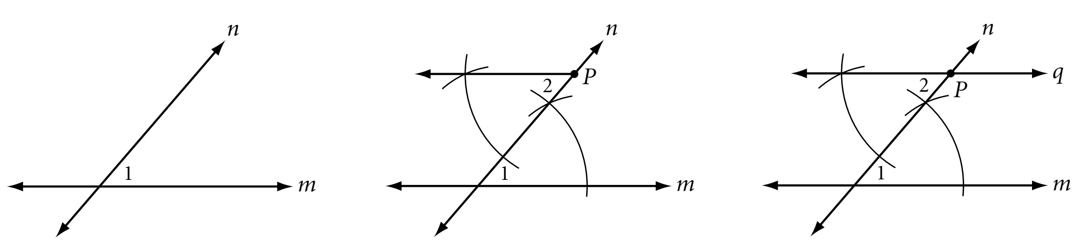
Investigation: Constructing Parallel Lines by Folding

Follow Steps 1–3 in your book to construct parallel lines with patty paper. Notice that the pairs of corresponding angles, alternate interior angles, and alternate exterior angles are congruent. (In this case, all are pairs of right angles.)

The next example shows another way to construct parallel lines.

EXAMPLE Use the Converse of the Alternate Interior Angles Conjecture to construct a pair of parallel lines. (Try to do this on your own before reading the solution.)

▶ ***Solution*** Draw two intersecting lines and label them *m* and *n*. Label one of the angles formed $\angle 1$. Label a point *P* on line *n*. Using point *P* as the vertex, duplicate $\angle 1$ on the opposite side of line *n*. Label the new angle $\angle 2$. Draw line *q* containing the new side of $\angle 2$.



Notice that line *m* and line *q* are cut by a transversal (line *n*), to form a congruent pair of alternate interior angles ($\angle 1$ and $\angle 2$). According to the converse of the AIA Conjecture, $m \parallel q$.

Now, see if you can use the converse of the Corresponding Angles Conjecture or the converse of the Alternate Exterior Angles Conjecture to construct a pair of parallel lines.

Write a summary of the construction methods you learned in this lesson.

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CONDENSED
LESSON
3.6 Construction Problems

In this lesson, you

- Construct polygons given information about some of the sides and angles

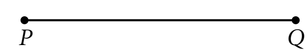
In this chapter, you have learned to construct congruent segments and angles, angle and segment bisectors, perpendiculars, perpendicular bisectors, and parallel lines. Once you know these basic constructions, you can create more advanced geometric figures.

Example A in your book shows you how to construct a triangle if you are given three segments to use as sides. This example also explores an important question: If you are given three segments, how many different-size triangles can you form? Read the example carefully.

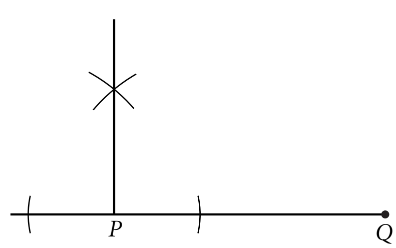
Example B shows you how to construct a triangle if you are given three angles. This example shows that three angles do *not* determine a unique triangle. Given three angle measures, you can draw an infinite number of triangles. The triangles will all have the same shape, but they will be different sizes.

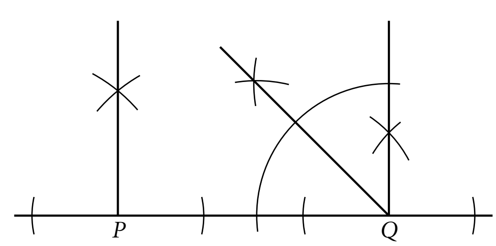
The examples below show some other constructions.

EXAMPLE A Using a compass and straightedge, construct $\triangle PQR$ with this segment as side $\overline{PQ}$ and with $m\angle P = 90°$ and $m\angle Q = 45°$.

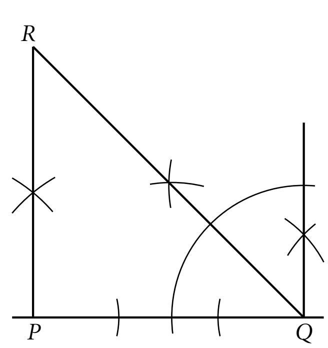


► ***Solution*** To construct $\angle P$, extend $\overline{PQ}$ to the left and construct a perpendicular to $\overleftrightarrow{PQ}$ through point P. To construct $\angle Q$, first construct a perpendicular to $\overleftrightarrow{PQ}$ through point Q. This creates a right angle with vertex Q. To create a 45° angle, bisect this angle.





To complete the construction, extend the sides of $\angle P$ and $\angle Q$ until they intersect. Label the intersection point R.



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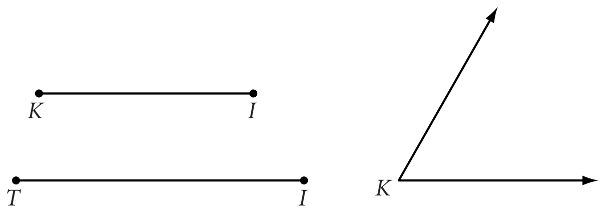
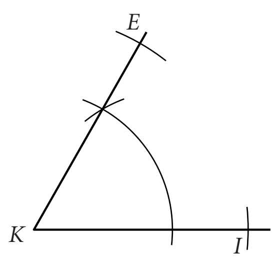
EXAMPLE B

Construct kite *KITE* with *KI* = *KE* and *TI* = *TE*, using the segments and angle below.

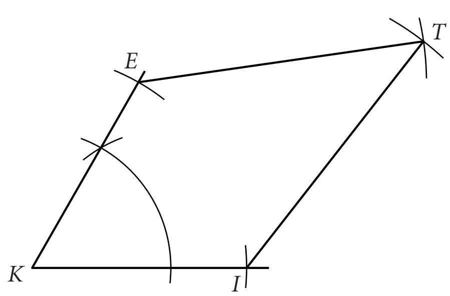


▶ ***Solution***

Copy $\overline{KI}$ and ∠*K*. Because *KI* = *KE*, copy $\overline{KI}$ on the other side of ∠*K* to create side $\overline{KE}$.



To locate vertex *T*, make a large arc with radius *TI* centered at point *I*. Vertex *T* must be on this arc. Because *TI* = *TE*, draw another large arc with radius *TI* centered at point *E*. The intersection of the two arcs is point *T*. Connect points *E* and *I* to point *T* to complete the kite.



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CONDENSED
LESSON
3.7

Constructing Points of Concurrency

In this lesson, you

- Construct the **incenter, circumcenter,** and **orthocenter** of a triangle
- Make conjectures about the properties of the incenter and circumcenter of a triangle
- **Circumscribe** a circle about a triangle and **inscribe** a circle in a triangle

You can use the constructions you learned in this chapter to construct special segments related to triangles. In this lesson, you will construct the angle bisectors and altitudes of a triangle, and the perpendicular bisectors of a triangle's sides. After you construct each set of three segments, you will determine whether they are *concurrent.* Three or more segments, lines, or rays are **concurrent** if they intersect in a single point. The point of intersection is called the **point of concurrency.**

Investigation 1: Concurrence

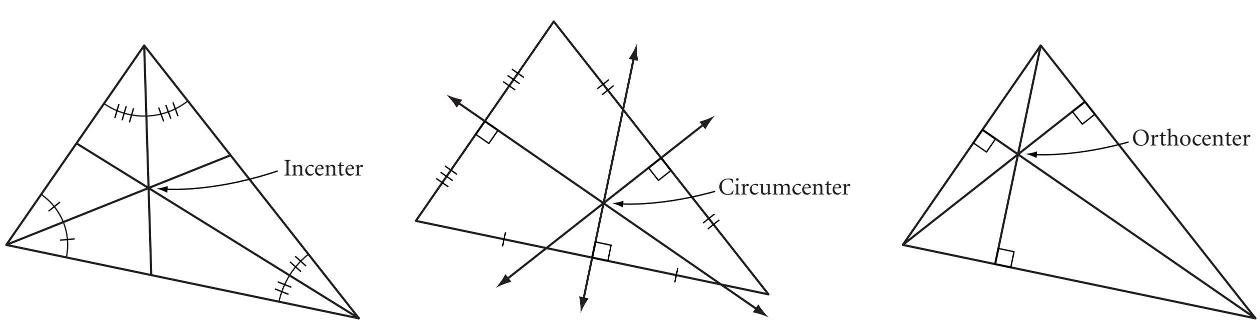
You can perform the constructions in this investigation with patty paper or with a compass and straightedge. Save your constructions to use in Investigation 2.

If you are using patty paper, draw a large acute triangle on one sheet and a large obtuse triangle on another. If you are using a compass, draw the triangles on the top and bottom halves of a piece of paper.

Construct the three angle bisectors of each triangle. You should find that they are concurrent. The point of concurrency is called the **incenter** of the triangle.

Start with two new triangles, one acute and one obtuse, and construct the perpendicular bisector of each side. You should find that, in each triangle, the three perpendicular bisectors are concurrent. The point of concurrency is called the **circumcenter.**

Finally, start with two new triangles and construct the altitude to each side. These segments are also concurrent. The point of concurrency is called the **orthocenter.**



Your observations in this investigation lead to the following conjectures.

Angle Bisector Concurrency Conjecture The three angle bisectors of a triangle are concurrent. **C-9**

(continued)

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Perpendicular Bisector Concurrency Conjecture The three perpendicular bisectors of a triangle are concurrent. **C-10**

Altitude Concurrency Conjecture The three altitudes (or the lines containing the altitudes) of a triangle are concurrent. **C-11**

For what type of triangle will the incenter, circumcenter, and orthocenter be the same point? If you don't know, experiment with different types of triangles (scalene, isosceles, equilateral, acute, obtuse, right).

Investigation 2: Incenter and Circumcenter

You will need your triangles from Investigation 1 for this investigation. Start with the two triangles for which you constructed the circumcenter. For each triangle, measure the distance from the circumcenter to each of the three vertices. Are the distances the same? Now, measure the distance from the circumcenter to each of the three sides. Are the distances the same? You can state your findings as the Circumcenter Conjecture.

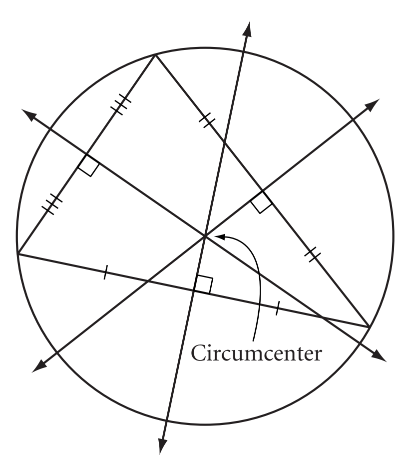
Circumcenter Conjecture The circumcenter of a triangle is equidistant from the three vertices. **C-12**

Now, start with the two triangles for which you constructed the incenter. Measure the distance from the incenter to each vertex. Then, measure the distance from the incenter to each side. What do you notice? You can summarize your findings in the following conjecture.

Incenter Conjecture The incenter of a triangle is equidistant from the three sides. **C-13**

Read the paragraph proof of the Circumcenter Conjecture on page 178 of your book and make sure you understand it.

Because the circumcenter is equidistant from the three vertices of a triangle, you can construct a circle centered at the circumcenter that passes through all three vertices. A circle that passes through each vertex of a polygon is **circumscribed** about the polygon.



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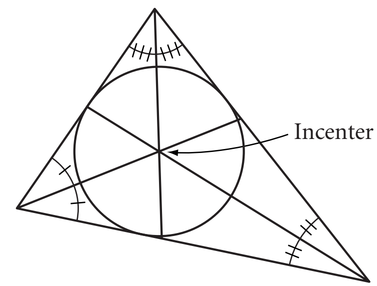
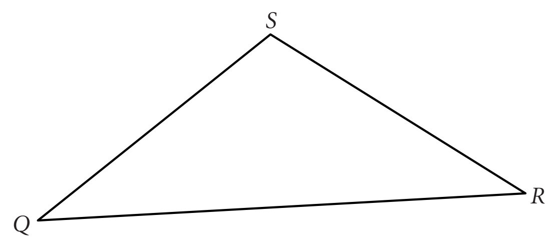


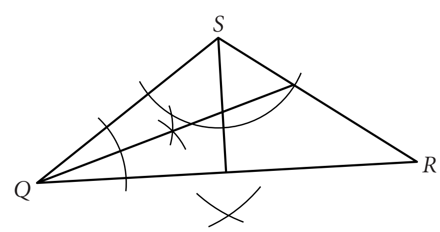
Now, read the paragraph proof of the Incenter Conjecture on page 178.

Because the incenter is equidistant from all three sides of a triangle, you can construct a circle centered at the incenter that is tangent to all three sides. A circle that is tangent to each side of a polygon is **inscribed** in the polygon.

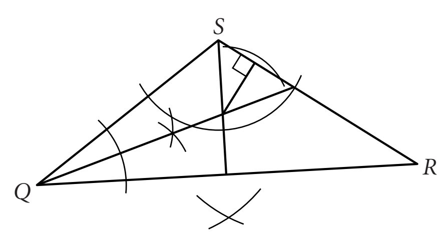
EXAMPLE Inscribe a circle in $\triangle QRS$.

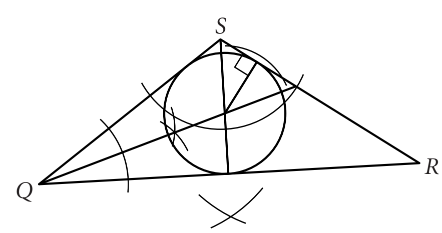


▶ ***Solution*** To find the center of the circle, construct the incenter. Note that you need to construct only two angle bisectors to locate the incenter. (Why?)



The radius of the circle is the distance from the incenter to each side. To find the radius, construct a perpendicular from the incenter to one of the sides. Here, we construct the perpendicular to $\overline{RS}$. Now, draw the circle.





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CONDENSED

LESSON

3.8 The Centroid

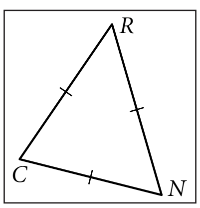
In this lesson, you

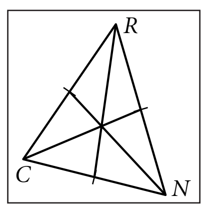
- Construct the **centroid** of a triangle
- Make conjectures about the properties of the centroid of a triangle

You have seen that the three angle bisectors, the three perpendicular bisectors of the sides, and the three altitudes of a triangle are concurrent. In this lesson, you will look at the three medians of a triangle.

Investigation 1: Are Medians Concurrent?

On a sheet of patty paper, draw a large scalene acute triangle and label it *CNR*. Locate the midpoints of the three sides and construct the medians. You should find that the medians are concurrent. Save this triangle.

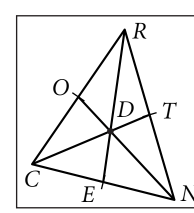




Now, start with a scalene obtuse triangle and construct the three medians. Are the medians concurrent? You can state your findings as a conjecture.

Median Concurrency Conjecture The three medians of a triangle are concurrent. **C-14**

The point of concurrency of the three medians is the **centroid.** On your acute triangle, label the medians $\overline{CT}$, $\overline{NO}$, and $\overline{RE}$. Label the centroid *D*.



Use your compass or patty paper to investigate the centroid: Is the centroid equidistant from the three vertices? Is it equidistant from the three sides? Is the centroid the midpoint of each median?

The centroid *D* divides each median into two segments. For each median, find the ratio of the length of the longer segment to the length of the shorter segment. You should find that, for each median, the ratio is the same. Use your findings to complete this conjecture.

Centroid Conjecture The centroid of a triangle divides each median into two parts so that the distance from the centroid to the vertex is ___________ the distance from the centroid to the midpoint of the opposite side. **C-15**

In Lesson 3.7, you learned that the circumcenter of a triangle is the center of the circumscribed circle and the orthocenter is the center of the inscribed circle. In the next investigation, you will discover a special property of the centroid.

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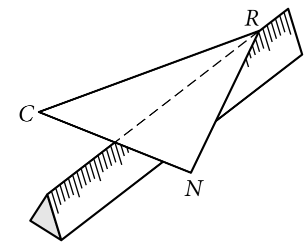
Investigation 2: Balancing Act

For this investigation, you will need a sheet of cardboard and your scalene acute triangle from Investigation 1.

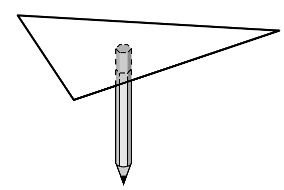
Place your patty-paper triangle on the cardboard. With your compass point, mark the three vertices, the three midpoints, and the centroid on the cardboard. Remove the patty paper and carefully draw the triangle and the medians on the cardboard. Cut out the cardboard triangle.



Try balancing the triangle by placing one of its medians on the edge of a ruler.

You should be able to get the triangle to balance. Repeat the process with each of the other medians. The fact that you can balance the triangle on each median means that each median divides the triangle into two triangular regions of equal area.

Now, try to balance the triangle by placing its centroid on the end of a pencil or pen. If you have traced and cut out the triangle carefully, it should balance. Because the triangle balances on its centroid, the centroid is the triangle's **center of gravity.**



You can state your findings as a conjecture.

Center of Gravity Conjecture The centroid of a triangle is the center of gravity of the triangular region. **C-16**

Notice that it makes sense that the triangle balances on the centroid because it balances on each median and the centroid is on each median. As long as the weight of the cardboard is distributed evenly throughout the triangle, you can balance any triangle on its centroid.

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